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On the time-dependent radiative transfer in photospheric plasmas: I. The analytical approach

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Abstract. This paper is the first in a series investigating time-dependent radiative transfer processes of x rays in photospheric plasmas at non-relativistic temperatures. Two parallel tracks are followed and each aspect is studied with an analytical and a Monte Carlo method. In this first paper, an analytical random walk model, describing the diffusion of photons in a homogeneous spherical geometry, is presented. The starting point is a set of moment equations derived earlier from the relativistic Boltzmann equation. Green functions for every individual scattering order are derived. The total Green function is the sum over all scattering orders and it solves the hyperbolic diffusion equation. An important element of flexibility arises from the introduction of the so-called characteristic parameter χ which is basically the inverse of the characteristic velocity. For high scattering orders, χ is equal to $\sqrt{3}$. By adjusting χ for the low scattering orders, agreement with numerical simulations is obtained. A full-scale quantitative discussion of the main results and their comparison with numerical simulations will be presented in the second paper of this series.

1. Introduction

This paper is the first of a series investigating time-dependent radiative transfer processes of x rays in non-degenerate ionised plasmas at non-relativistic temperatures $kT < m_e c^2$. In doing so we follow two parallel tracks, studying each aspect both analytically and with a Monte Carlo method. The content of this paper is purely analytical and may be of interest to both physicists and mathematicians. Addressing the physicists we present first a few words about the motivation for developing the theory presented here and to be elaborated on in subsequent papers.

The observations in x-ray and gamma-ray astronomy during the past years have increasingly demonstrated the importance of time-dependent transport processes in astrophysical objects. Most remarkable are the so-called x-ray bursters which exhibit luminosity profiles varying in intensity and spectrum over time scales between minutes to a few thousand seconds. In the case of the so-called 1700 s event reported by Hoffman *et al* (1978), a precursor pulse (lasting about 4 s) is neatly separated from the main event (lasting longer than 1000 s) by a 3 s period of silence during which no x-ray luminosity was detected in the 1.5-43 keV band. Precursors have also been observed in the luminosity profiles of other x-ray transients. Lewin *et al* (1984) and Tawara *et al* (1984) have suggested that precursor pulses can possibly be explained in terms of

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rapidly expanding neutron star photospheres. The rapid expansion, caused by a burst reaching or exceeding the Eddington limit, is accompanied by a drop of the matter temperature in the plasma layers above the neutron star's normal radius, i.e. above $R_n \simeq 10$ km. According to Lewin *et al* (1984) and Tawara *et al* (1984), all x-ray frequencies are shifted down to energy channels below 1.5 keV when passing through the cooled plasma, thus explaining the silence in the detector range. The subsequent slow contraction of the photosphere is then accompanied by a rising temperature and the emission of the main x-ray burst over an extended time period. This picture appears to be simple and is certainly convenient from the astronomer's point of view. However, it was pointed out earlier by Schweizer (1985a) that the realisation of this picture as a detailed theoretical model would be enormously complicated. The main source of the difficulties is given by the *photospheric* properties of the plasma layers above R_n. By photospheric we mean that the radiative transfer is largely dominated by scatterings, i.e. a photon will, on the average, undergo a random walk through the plasma and escape before being absorbed. This still allows for significant thermal contact between matter and radiation such as Comptonisation combined with weak emission and absorption. The order of magnitude considerations in Schweizer (1985a) suggest, however, that this would be insufficient to establish local thermal equilibrium between matter and radiation in most parts of the plasma shell above R_n. In addition one is faced with photon mean free paths comparable to the size of the overall geometry. Since such situations preclude the applicability of standard techniques for moving atmospheres the investigation of new methods is required.

In this paper we develop and present an analytical theory describing the random walk of photons in a homogeneous spherical geometry with radius R. The sources can be situated in the centre of the sphere (point sources) or on a shell with radius $r_0 \leqslant R$. General spherical sources are obtained from linear superpositions of shell-like sources. The case of semi-infinite geometry is included in the limit of R approaching infinity, where we put $r_0 = R - \tau_0$ with τ_0 denoting the optical depth of the source below the boundary.

The diffusion profiles are given for each individual scattering order, i.e. we treat the total luminosity profile as the sum of the N-times scattered photons. Thermal couplings between photons and electrons are not yet discussed in this paper. But let us point out here that Comptonisation can easily be included by adding up the individual specific intensities produced by each scattering order. For a given scattering order the specific intensity is determined by the number of scatterings, the electron temperature and the spectrum of the source. This will allow for the modelling of the spectral evolution of a burst.

The starting point in our approach is a set of hyperbolic diffusion equations derived earlier from the relativistic transfer equation. The theory elaborated here is an extended and improved version of a random walk model presented in an earlier paper (Schweizer 1984a, paper I). The scatterings between photons and electrons are treated as elastic and coherent, i.e. we allow for Thomson scattering only. Furthermore, our random walk model idealises the collisions as isotropic. A comparison with results from numerical simulations shows that this idealisation is justified. The content of this paper is technical. In an attempt to keep the size of this paper reasonable, we decided to postpone a full-scale quantitative discussion of the main results and the comparison with the Monte Carlo simulations to the next paper of this series; see Schultz and Schweizer (1987). The applicability of the formalism developed here and the agreement with Monte Carlo is, however, briefly discussed at the end of this paper.

This paper is organised as follows. The basic concepts are developed in § 2 for the simple case of an infinite medium. Section 2 is easy to read and it is possible to skip from here to § 5 where the main results are summarised for the case of finite spherical geometries. Sections 3 and 4 are technical and some lengthy intermediate calculations have been omitted. In § 3, we establish our random walk model for the case of a spherical geometry and show how it can be reduced to the case of a two-sided slab geometry in only $(1+1)$ dimensions. In § 4, we expand the generating function, derived in § 3, in an infinite series of modified Bessel functions and extract from this expansion explicit expressions for the individual scattering orders. In appendix 1, we show how our random walk model can be modified for the case of an absorptive medium.

2. Diffusion in an infinite medium

The radiative plasmas considered in this paper are characterised by low densities and high temperatures. The lifetime t_a of a photon with respect to absorption is typically large in this case; see for instance the numbers in table 2 in Schweizer (1984b). It is therefore conceivable that for a given geometry the average travel time of a photon through the plasma may turn out to be shorter than t_a, thus defining a radiative transfer process dominated essentially by elastic collisions, i.e. Compton scattering. The main focus of this paper is on the random walk aspect. The thermal couplings between photons and electrons are ignored at this stage. Thermal couplings and spectral evolutions of diffusion profiles will be discussed in the third paper of this series; see, however, appendix 1.

Starting from the relativistic Boltzmann equation, one can derive a hierarchy of frequency integrated moment equations for the photon number density N, the photon 3-current $\boldsymbol{J}$, and all the higher order moments; see Schweizer (1985b). The first two of these equations may be considered as hyperbolic diffusion equations and are, for the case of a non-absorptive plasma cloud moving slowly in a flat spacetime, given by

$$\partial N/\partial t+\nabla\cdot\boldsymbol{J}=S \tag{1}$$

$$\kappa_T\boldsymbol{J}+(\partial/\partial t)\boldsymbol{J}+\tfrac{1}{3}\nabla N=0. \tag{2}$$

The term $S(t,\boldsymbol{x})$ in (1) stands for a photon source inside the cloud and $\kappa_T=n_e\sigma_T$ is the Thomson opacity (n_e denoting the electron number density and σ_T the Thomson cross section). Let us further assume that n_e is constant everywhere and make the identifications $t\equiv t\cdot\kappa_T$ and $\boldsymbol{x}\equiv\boldsymbol{x}\cdot\kappa_T$. The general case with variable n_e will be investigated in one of the subsequent papers. Differentiating (1) and (2) with respect to t and $\boldsymbol{x}$, one can easily separate the variables N and $\boldsymbol{J}$. This yields the following second-order differential equation for N:

$$\mathscr{L}N\equiv\frac{\partial}{\partial t}N+\frac{\partial^2}{\partial t^2}N-\tfrac{1}{3}\Delta N=S+\frac{\partial}{\partial t}S. \tag{3}$$

The solution of (3) is, modulo homogeneous solutions, given by

$$N=\bar{G}*S \tag{4}$$

where

$$\bar{G}\equiv(1+\partial_t)G$$

and

$$G = \frac{1}{(2\pi)^4} \int d\omega \int d^3p \frac{\exp(-i\omega t)\exp(-i\boldsymbol{p}\cdot\boldsymbol{x})}{\frac{1}{3}\boldsymbol{p}^2 - \omega^2 - i\omega}. \tag{5}$$

The * symbol stands for a spacetime convolution integral. Expression (5) is a fundamental solution of $\mathscr{L}$ and it is unique as a tempered distribution with support on the half-space $t \geqslant 0$.

Equation (3), known as the so-called 'telegraphy equation', is a wave equation and diffusion equation in one. The characteristic velocity is given by $v_c = 1/\sqrt{3}$. This value is reasonable for the description of isotropically distributed photon momenta and it can be explained as follows. The local photon velocity is equal to unity. Since there are no privileged spatial directions the root mean square of the velocity in a given direction is equal to $1/\sqrt{3}$. This argument breaks down if it comes to the description of photons having suffered few collisions with electrons. For unscattered photons the characteristic velocity is equal to unity. We introduce, for this reason, an extra element of flexibility and replace $v_c^2 = \frac{1}{3}$ in (3) by χ^{-2}, where the so-called *characteristic parameter* χ is allowed to take values between unity and $\sqrt{3}$. It is shown later in this section how one has to deal with χ in a concrete situation.

It is our aim to write the Green functions as the sum of all scattering orders as follows:

$$\bar{G}(t, \boldsymbol{x}; \chi) = \sum_{N=1}^{\infty} \bar{W}_N(t, \boldsymbol{x}; \chi) \qquad \text{or} \qquad G(t, \boldsymbol{x}; \chi) = \sum_{N=1}^{\infty} W_N(t, \boldsymbol{x}; \chi) \tag{6}$$

where $\bar{W}_N = (1 + \partial_t) W_N$, and $\bar{P}_{N-1} \equiv \bar{W}_N * S$ or $P_{N-1} = W_N * S$ describe the temporal profiles of the $(N-1)$-times scattered photons reaching the spacetime volume element at $(t, \boldsymbol{x})$ after being released from the source S. The constituents $\bar{W}_N$ are *unique* if we interpret them as the N-fold convolution products

$$\bar{W}_N = \underbrace{\bar{W}_1 * \ldots * \bar{W}_1}_{N\ \text{factors}} \tag{7}$$

of a *random distribution* $\bar{W}_1$ describing single and collision-free displacements. The Fourier transform of (7) is given by

$$\int dt \int d^3x \exp(i\omega t)\exp(i\boldsymbol{p}\cdot\boldsymbol{x})\,\bar{W}_N = \tilde{\bar{W}}_N = \tilde{\bar{W}}_1^N. \tag{8}$$

The sum over N leads to a geometric series, the result of which must be equal to $(1 - i\omega)$ times the denominator in the integral (5) with χ^{-2} in place of $\frac{1}{3}$, i.e.

$$\tilde{\bar{G}} = \sum_{N=1}^{\infty} \tilde{\bar{W}}_N = \frac{\tilde{\bar{W}}_1}{1 - \tilde{\bar{W}}_1} = \frac{1 - i\omega}{\chi^{-2}p^2 - \omega^2 - i\omega}. \tag{9}$$

Solving this equation for W_1 yields

$$W_1 = \frac{1 - i\omega}{\chi^{-2}p^2 + (1 - i\omega)^2} \tag{10}$$

or

$$W_1(t, \boldsymbol{x}; \chi) = \frac{\chi^2}{4\pi r} e^{-t}\delta(t - \chi r) \tag{11}$$

where $r \equiv |\boldsymbol{x}|$; see also paper I in this context. Expressions for $\bar{W}_N$ and W_N can now be given as Fourier representations or explicitly in integrated form as

$$W_N(t, r; \chi) = \frac{\chi^2}{4\pi r} \mathrm{e}^{-t} \frac{(t/2)^{N-1}}{(N-1)!} \delta(t - \chi r)$$
$$+ \frac{(\frac{1}{2})^N \chi^3}{8\pi} \mathrm{e}^{-t} \sum_{\nu=3}^{N \geq 3} \frac{(\nu-1) EV(\nu-1)}{[((\nu-1)/2)!]^2} \frac{t^{N-\nu}}{(N-\nu)!} \left(\frac{t^2 - \chi^2 r^2}{4}\right)^{(\nu-3)/2} \theta(t - \chi r) \tag{12}$$

where

$$EV(n) = \begin{cases} 1 & \text{for } n = 0, 2, 4, 6, \ldots \\ \text{zero} & \text{for all other cases.} \end{cases}$$

Expression (12) is included as a special case in the main results derived below; see (50) for the case of R approaching infinity.

Starting from the Fourier representations, it is easy to see that the following normalisation conditions hold:

$$\int \mathrm{d}t \int \mathrm{d}^3x \bar{W}_N = 1 \qquad \text{and} \qquad \int \mathrm{d}t \int \mathrm{d}^3x\, W_N = 1.$$

The measures $\bar{W}_N\, \mathrm{d}t\, \mathrm{d}^3x$ are, on the other hand, not positive. Furthermore, we want to point out here that the positivity may also be violated for the measures $W_N\, \mathrm{d}t\, \mathrm{d}^3x$ once boundary terms are added to the expression (12). In spite of this we proceed as if these highly singular functions are nice probability measures, being at the same time careful with the interpretation of the results. Whenever we encounter profiles $\bar{P}_N$ or P_N taking negative values for certain times t, we interpret this as 'no signal' and put $\bar{P}_N$ or P_N equal to zero. It is plausible that the singular character and non-positivity of these measures disperses for smooth sources. Pathologies are most likely to become apparent when the $\bar{W}_N$ and W_N are convoluted with singular point-like sources of the form

$$S(t, \boldsymbol{x}) = \sigma(t) \delta^3(\boldsymbol{x}). \tag{13}$$

We shall nevertheless include sources of this type in our analysis. A comparison with results from numerical simulations suggests, however, that the temporal diffusion profiles $P_N(t, r; \chi_N)$ provide better descriptions than the $\bar{P}_N$. The characteristic parameters χ_N have to be suitably chosen for every individual scattering order. A good choice is $\chi_0 = 1$, $\chi_1 = 1.05$, $\chi_2 = 1.1$, $\chi_3 = 1.15$, $\chi_4 = 1.2$, $\chi_5 = 1.3$, $\chi_6 = 1.4$, $\chi_7 = 1.5$, $\chi_8 = 1.6$ and $\chi_N = \sqrt{3}$ for $N \geq 9$.

Starting from the respective Fourier representation, it is easy to see that

$$\int \mathrm{d}^3x\, \bar{W}_{N+1} = \frac{t^N}{N!} \mathrm{e}^{-t} \tag{14}$$

and

$$\int \mathrm{d}^3x\, W_{N+1} = \frac{t^{N+1}}{(N+1)!} \mathrm{e}^{-t}. \tag{15}$$

These functions describe the probabilities that a photon, released at time zero, has suffered N scatterings at time t and this independently of its position. Since these distribution functions should clearly be Poissonian as is the case for (14), we conclude that from the statistical viewpoint the $\bar{P}_N$ profiles are the better ones. However, better agreement with Monte Carlo simulations is obtained with the P_N profiles, and we shall, for this reason, proceed with the P_N. This is equivalent to deleting the $\dot{S}$ term in (3). This term is negligible for sources varying slowly over one time unit t_T.

The effects of absorption are so far ignored. It is, however, shown in appendix 1 how the theory developed in this paper can be generalised for the case of absorptive media.

3. Diffusion in a finite spherical geometry and the reduction to a (1 + 1)-dimensional problem

In this section we consider the emission of photons from shell-like sources of the type

$$S(t, x) = \delta(r - r_0)\mathscr{A}(t). \tag{16}$$

The shells with radius r_0 are placed inside a plasma ball

$$B_R = \{x \in R^3 \,|\, |x| = r \leqslant R\}.$$

For the sake of simplicity, we temporarily put $\chi = 1$ for all scattering orders. We shall restore χ in § 5 below. In addition, we proceed with $\mathscr{A}(t) = \delta(t)$, i.e. we establish functions which later have to be convoluted with general expressions for $\mathscr{A}(t)$.

The profile for the first displacement is given by

$$\bar{W}_1^R(t, r, r_0) = \bar{W}_1 *_R S = (1 + \partial_t) W_1 *_R S$$
$$= (1 + \partial_t)\frac{e^{-t}}{4\pi}\int_{B_R} d^3x' \,\frac{\delta(t - |x - x'|)}{|x - x'|}\delta(|x'| - r_0).$$

The profile for the $(N + 1)$th displacement is given by

$$\bar{W}_{N+1}^R(t, r, r_0) = \bar{W}_1 *_R \bar{W}_N^R$$
$$= (1 + \partial_t)\frac{1}{4\pi}\int_{-\infty}^{\infty} dt' \exp[-(t - t')]$$
$$\times \int_{B_R} d^3x' \,\frac{\delta(t - t' - |x - x'|)}{|x - x'|}\bar{W}_N^R(t', |x'|, r_0). \tag{17}$$

Note that only those photons contribute to $\bar{W}_{N+1}^R(t, r, r_0)$ which have never moved beyond the boundary at $r = R$. In proceeding like this, $\bar{W}_{N+1}^R$ can equivalently be interpreted as the profile created by those N-fold scattered photons which have been released from a shell source in an *infinite* medium but have never moved past the point $r = R$.

A straightforward calculation shows that

$$\bar{W}_{N+1}^R(t, r, r_0)$$
$$= \frac{r_0}{r}\left(\frac{1}{2}\right)^{N+1} e^{-t}\int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_N \int_{-R}^{R} dr_1 \ldots \int_{-R}^{R} dr_N \,\delta(t - t_1 - |r - r_1|)$$
$$\times \delta(t_1 - t_2 - |r_1 - r_2|) \ldots \delta(t_{N-1} - t_N - |r_{N-1} - r_N|)$$
$$\times [\delta(t_N - |r_N - r_0|) - \delta(t_N - |r_N + r_0|)]. \tag{18}$$

The profiles for point-like sources are easily obtained as follows: divide (16) by $4\pi r_0^2$ and take the limit as r_0 approaches zero. Since $r_0^{-1}\bar{W}_N^R(t, r, r_0)$ converges to zero as r_0 decreases to zero, the profiles for a point source are given by

$$\bar{W}_N^R(t, r) = \frac{1}{4\pi} \frac{\partial}{\partial r_0}\bigg|_0 \left(\frac{1}{r_0} \bar{W}_N^R(t, r, r_0)\right). \tag{19}$$

This result can also be obtained directly by performing all convolution integrals with (13) in place of (16).

Our main goal is the explicit computation of the integrals in (18). The key to the solution of this problem is the close connection of (18) to a tractable random walk process in only one time plus one space dimension. For this case let x denote the coordinate along the space axis. It has been demonstrated earlier in paper I that a distinct random distribution, describing the propagation of massless particles in an opaque medium in the $(1+1)$-dimensional Minkowski space, is given by

$$\bar{w}_1(t, x) = \tfrac{1}{2}\delta(t - |x|)\, \mathrm{e}^{-t}. \tag{20}$$

Let us consider the geometry of a two-sided slab covering the interval $[-R, R]$ and containing a point source

$$S(t, x) = \delta(x - x_0)\delta(t) \tag{21}$$

placed at an arbitrary position $x_0 \in [-R, R]$. Extending all spatial convolution integrals over the slab only, it follows easily that the profile of the N-fold scattered photons arriving at x is given by

$$\bar{w}_{N+1}^R(t, x, x_0) = \left(\frac{1}{2}\right)^{N+1} \mathrm{e}^{-t} \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \int_{-\infty}^{\infty} \mathrm{d}t_N \int_{-R}^{R} \mathrm{d}x_1 \ldots \int_{-R}^{R} \mathrm{d}x_N\, \delta(t - t_1 - |x - x_1|)$$
$$\times\, \delta(t_1 - t_2 - |x_1 - x_2|) \ldots \delta(t_{N-1} - t_N - |x_{N-1} - x_N|)\delta(t_N - |x_N - x_0|). \tag{22}$$

A comparison with (18) shows that we have the following simple relation between the $(1+1)$-dimensional case and the $(1+3)$-dimensional spherical case;

$$\bar{W}_{N+1}^R(t, r, r_0) = (r_0/r)[\bar{w}_{N+1}^R(t, r, r_0) - \bar{w}_{N+1}^R(t, r, -r_0)]. \tag{23}$$

The solution of our problem reduces thus to the explicit computation of (22) which can be written in compact form as

$$\bar{w}_{N+1}^R(t, x, x_0) = (\tfrac{1}{2})^{N+1}\, \mathrm{e}^{-t} \int_{-R}^{R} \mathrm{d}x_1 \ldots \int_{-R}^{R} \mathrm{d}x_N\, \delta(t - |x - x_1| - \ldots - |x_N - x_0|). \tag{24}$$

It is advantageous to go to the temporal Fourier transform of (24) and proceed with

$$\tilde{\bar{w}}_{N+1}^R(\omega, x, x_0) = (\tfrac{1}{2})^{N+1} \int_{-R}^{R} \mathrm{d}x_1 \ldots \int_{-R}^{R} \mathrm{d}x_N$$
$$\times \exp[-(1 - \mathrm{i}\omega)|x - x_1| - \ldots - (1 - \mathrm{i}\omega)|x_N - x_0|]. \tag{25}$$

For parameters $0 \leq \varepsilon \leq 1$, we now define a *generating function* $\bar{g}_\varepsilon^R$ in terms of its Fourier transform as follows:

$$\tilde{\bar{g}}_\varepsilon^R(\omega, x, x_0) \equiv \sum_{N=0}^{\infty} \varepsilon^{N+1} \tilde{\bar{w}}_{N+1}^R(\omega, x, x_0). \tag{26}$$

This implies that

$$\tilde{w}^{R}_{N+1}(\omega, x, x_0) = \frac{1}{(N+1)!}\left(\frac{\partial}{\partial\varepsilon}\right)^{N+1}_{\Big|0} \tilde{g}^{R}_{\varepsilon}(\omega, x, x_0). \tag{27}$$

The generating function (26) is easily recognised as the Neumann series solving the integral equation

$$\tilde{G}(\omega, y) = \tfrac{1}{2}\varepsilon \exp[-(1-\mathrm{i}\omega)|y-x_0|] + \int_{-R}^{R} \mathrm{d}y_1\, \tfrac{1}{2}\varepsilon \exp[-(1-\mathrm{i}\omega)|y-y_1|]\tilde{G}(\omega, y_1) \tag{28}$$

i.e.

$$\tilde{G}(\omega, y) = \tilde{g}^{R}_{\varepsilon}(\omega, y, x_0).$$

Let us define $G(\omega, x) \equiv \tilde{G}(\omega, x+x_0)$ and rewrite (28) as

$$G(\omega, x) = \tfrac{1}{2}\varepsilon \exp[-(1-\mathrm{i}\omega)|x|] + \int_{-R-x_0}^{R-x_0} \mathrm{d}x_1\, \tfrac{1}{2}\varepsilon \exp[-(1-\mathrm{i}\omega)|x-x_1|]G(\omega, x_1). \tag{29}$$

It is straightforward to solve equation (29) with the standard techniques presented in the first chapter of Sobolev's (1963) book. The principal steps are as follows. First we split the integral in (29) into a sum $(\varepsilon/2)(I_1+I_2) = (\varepsilon/2)\Sigma$, where

$$I_1(\omega, x) = \int_{-R-x_0}^{x} \exp[-(1-\mathrm{i}\omega)|x-x_1|]G(\omega, x_1)\, \mathrm{d}x_1$$

and

$$I_2(\omega, x) = \int_{x}^{R-x_0} \exp[-(1-\mathrm{i}\omega)|x-x_1|]G(\omega, x_1)\, \mathrm{d}x_1.$$

The usual manipulations show that Σ is subject to the following differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Sigma - a^2\Sigma = -\kappa\varepsilon \exp(-\kappa|x|) \tag{30}$$

where $\kappa \equiv 1-\mathrm{i}\omega$ and $a^2 \equiv \kappa(\kappa-\varepsilon)$. It is easy to determine the general form of $(\varepsilon/2)\Sigma$ from (30) and we substitute it into the right-hand side of (29), thus finding the following general expression for G:

$$G(\omega, x) = \frac{\varepsilon\kappa}{2a}\exp(-a|x|) + \frac{\varepsilon}{2}A\, \mathrm{e}^{ax} + \frac{\varepsilon}{2}B\, \mathrm{e}^{-ax}. \tag{31}$$

The two constants A and B are undetermined at this point. Since (29) must hold for arbitrary $x \in [-R-x_0, R-x_0]$, we may choose convenient values for x, e.g. $x=0$, $x=-R_0-x_0$, $x=R_0-x_0$, and substitute (31) for at least two of these choices into (29). This provides two equations for the two unknowns A and B. It is, however, advantageous to work with three equations. A straightforward calculation shows that

$$\begin{aligned} \tilde{g}^{R}_{\varepsilon}(\omega, x, x_0) = {} & \frac{\varepsilon}{2q}\exp(-\kappa q|x-x_0|)\frac{\{1+[(1-q)/(1+q)]^2\exp[-2\kappa q(2R-|x-x_0|)]\}}{\{1-[(1-q)/(1+q)]^2\exp(-4\kappa qR)\}} \\ & -\frac{\varepsilon}{2q}\left(\frac{1-q}{1+q}\right)\frac{\{\exp[-\kappa q(2R-x-x_0)]+\exp[-\kappa q(2R+x+x_0)]\}}{\{1-[(1-q)/(1+q)]^2\exp(-4\kappa qR)\}} \end{aligned} \tag{32}$$

where

$$q \equiv \left(\frac{\kappa-\varepsilon}{\kappa}\right)^{1/2}.$$

Finally, we want to point out that

$$\bar{g}_\varepsilon^R(t, x, x_0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \bar{\bar{g}}_\varepsilon^R(\omega, x, x_0) \exp(-i\omega t) \tag{33}$$

is a generalisation of the Green function $\bar{g}$ solving the $(1+1)$-dimensional telegraphy equation

$$\left(\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\bar{g} = (1+\partial_t)\delta(x-x_0)\delta(t) \tag{34}$$

or, more specifically, $\bar{g} = \bar{g}_{\varepsilon=1}^R$. Furthermore, $\bar{g}_\varepsilon^R$ satisfies the following boundary conditions:

$$\left(\frac{\partial}{\partial t} \pm \frac{\partial}{\partial x} + 1\right)_{x=\pm R} \bar{g}_\varepsilon^R = 0. \tag{35}$$

This is a direct consequence of (28) and (33). Taking the sum in (23) over all indices N we see that (34) together with (26) generalises to

$$\mathscr{L} \sum_{N=0}^{\infty} \bar{W}_{N+1}^R(t, r, r_0) = (1+\partial_t)\delta(r-r_0)\delta(t). \tag{36}$$

This means that

$$\bar{G}^R(t, r, r_0) \equiv \sum_{N=0}^{\infty} \bar{W}_{N+1}^R(t, r, r_0)$$

is the Green function of the $(1+3)$-dimensional telegraphy equation for the case of a shell source $r_0 \leq R$. This concludes our conceptual discussion of the spherical $(1+3)$-dimensional case and its reduction to a $(1+1)$-dimensional problem.

4. The generating function as a series of modified Bessel functions

Our next goal is the explicit calculation of the Fourier transform of (32). In their analysis of the one-sided slab in $(1+1)$ dimensions with source at the origin, Nagel and Mészáros (1985) are faced with the same problem for the case of a different expression for $\bar{\bar{g}}$. As they point out there it is, in principle, straightforward to determine the poles and residues of an expression like (32). Such a procedure was also attempted by Minin (1971) in a similar context. Nagel and Mészáros show, however, that it is far more advantageous to proceed with geometric series expansions and Fourier transformation of each term of the series.

It is easy to see that (32) can be expanded as follows:

$$\begin{aligned}
\bar{\bar{g}}_\varepsilon(\omega, x, x_0) = \varepsilon \sum_{n=0}^{\infty} \Bigg(& [(1-q)/(1+q)]^{2n} \exp[-\kappa q(4nR + |x-x_0|)] \\
& + \left(\frac{1-q}{1+q}\right)^{2(n+1)} \exp\{-\kappa q[4(n+1)R - |x-x_0|]\}\Bigg)(2q)^{-1} \\
& - \varepsilon \sum_{n=0}^{\infty} \frac{[(1-q)/(1+q)]^{2n+1}}{2q} (\exp\{-\kappa q[2(2n-1)R - x - x_0]\} \\
& + \exp\{-\kappa q[2(2n+1)R + x + x_0]\}).
\end{aligned} \tag{37}$$

Define

$$Y_1(n) \equiv 4nR + |x - x_0| \qquad Y_2(n) \equiv 4(n+1)R - |x - x_0|$$
$$Y_3(n) \equiv 2(2n+1)R - x - x_0 \qquad Y_4(n) \equiv 2(2n+1)R + x + x_0.$$

The Fourier transformation of (37) is then given by integrals of the standard form

$$(1+\partial_t)\hat{J}(M;\, Y_j(n)) \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} \exp(-\mathrm{i}\omega t)\left(\frac{1-q}{1+q}\right)^M \frac{\exp(-\kappa q Y_j(n))}{2q}. \tag{38}$$

It is shown in appendix 2 that

$$\hat{J}(M, Y_j(n)) = \tfrac{1}{2}\theta(t - Y_j(n)) \exp[-(1-\varepsilon/2)t] I_M\left(\frac{\varepsilon}{2}(t^2 - Y_j^2(n))^{1/2}\right)\left(\frac{t - Y_j(n)}{t + Y_j(n)}\right)^{M/2} \tag{39}$$

where

$$I_M(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{M+2k} z^{M+2k}}{k!\,\Gamma(M+k+1)} \tag{40}$$

is the modified Bessel function of the first kind of order M. For a sketchy derivation of (39), the reader is also referred to Nagel and Mészáros (1985).

The generating function can now be written as

$$\bar{g}_\varepsilon(t, x, x_0) = (1+\partial_t) g_\varepsilon(t, x, x_0) \tag{41}$$

where

$$\begin{aligned}
g_\varepsilon(t, x, x_0) = \frac{\varepsilon}{2} &\exp[-(1-\varepsilon/2)t] \\
&\times \Bigg[\sum_{n=0}^{\infty} \theta(t - 4nR - |x - x_0|) \\
&\times I_{2n}\left(\frac{\varepsilon}{2}[t^2 - (4nR + |x - x_0|)^2]^{1/2}\right)\left(\frac{t - 4nR - |x - x_0|}{t + 4nR + |x - x_0|}\right)^n \\
&+ \sum_{n=0}^{\infty} \theta[t - 4(n+1)R + |x - x_0|] \\
&\times I_{2(n+1)}\left(\frac{\varepsilon}{2}\{t^2 - [4(n+1)R - |x - x_0|]^2\}^{1/2}\right)\left(\frac{t - 4(n+1)R + |x - x_0|}{t + 4(n+1)R - |x - x_0|}\right)^{n+1} \\
&- \sum_{n=0}^{\infty} \theta[t - 2(2n+1)R + x + x_0] \\
&\times I_{2n+1}\left(\frac{\varepsilon}{2}\{t^2 - [2(2n+1)R - x - x_0]^2\}^{1/2}\right) \\
&\times \left(\frac{t - 2(2n+1)R + x + x_0}{t + 2(2n+1)R - x - x_0}\right)^{(2n+1)/2} \\
&- \sum_{n-0}^{\infty} \theta[t - 2(2n+1)R - x - x_0] \\
&\times I_{2n+1}\left(\frac{\varepsilon}{2}\{t^2 - [2(2n+1)R + x + x_0]^2\}^{1/2}\right) \\
&\times \left(\frac{t - 2(2n+1)R - x - x_0}{t + 2(2n+1)R + x + x_0}\right)^{(2n+1)/2}\Bigg].
\end{aligned} \tag{42}$$

The Green functions solving the telegraphy equation for the various geometries discussed in this paper follow directly from (42) for $\varepsilon = 1$ and the discussion at the end of the previous section. The individual scattering orders $\bar{w}_N^R = (1+\partial_t)w_N^R$ can now be extracted from (42) by systematically differentiating this expression with respect to ε as indicated in (27).

For $r \geqslant 0$ and $r_0 \geqslant 0$ we define

$$D_N^R(t, r, r_0) \equiv w_N^R(t, r, r_0) - w_N^R(t, r, -r_0). \tag{43}$$

A lengthy but straightforward calculation shows that

$$D_N^R(t, r, r_0) = \Delta_N^R(t; r-r_0) - \Delta_N^R(t; r+r_0) \tag{44}$$

where

$$\begin{aligned}
\Delta_N^R(t; z) = {} & (\tfrac{1}{2})^N \, e^{-t} \sum_{\nu=1}^{N} \frac{EV(\nu-1)}{[((\nu-1)/2)!]^2} \frac{t^{N-\nu}}{(N-\nu)!} \left(\frac{t^2 - z^2}{4}\right)^{(\nu-1)/2} \theta(t-|z|) \\
& + (\tfrac{1}{2})^N \, e^{-t} \sum_{\nu=2}^{N\geqslant 2} \frac{t^{N-\nu}}{(N-\nu)!} \sum_{n=1}^{\nu-1} \frac{EV(\nu-n-1)}{[(\nu-n-1)/2]![(\nu+n-1)/2]!} \\
& \times \left[\left(\frac{t^2-(2nR-|z|)^2}{4}\right)^{(\nu-n-1)/2} \left(\frac{t-2nR+|z|}{2}\right)^n \theta(t-2nR+|z|) \right. \\
& \left. + \left(\frac{t^2-(2nR+|z|)^2}{4}\right)^{(\nu-n-1)/2} \left(\frac{t-2nR-|z|}{2}\right)^n \theta(t-2nR-|z|) \right].
\end{aligned} \tag{45}$$

Since

$$\frac{\partial}{\partial r_0}\Big|_0 \Delta_N^R(t; r-r_0) = -\frac{\partial}{\partial r} \Delta_N^R(t; r)$$

we have

$$\frac{\partial}{\partial r_0}\Big|_0 D_N^R(t; r, r_0) = -2\frac{\partial}{\partial r} \Delta_N^R(t; r). \tag{46}$$

Together with (19), (23) and (43), the following compact expression for the scattering orders of a point-like source is obtained:

$$W_N^R(t, r) = -\frac{1}{2\pi r} \frac{\partial}{\partial r} \Delta_N^R(t; r). \tag{47}$$

This concludes the technical part of our paper and we can go ahead and restore the characteristic parameter χ for general values.

5. Summary of the main results and conclusions

We start this section by summarising the main results for general values of χ. It is straightforward to rederive (18) for arbitrary χ and establish the following relation:

$$\bar{W}_{N+1}^R(t, r, r_0; \chi) = \chi \bar{W}_{N+1}^{\chi R}(t, \chi r, \chi r_0).$$

Recalling (23) and (43), we see that

$$\bar{W}_{N+1}^R(t, r, r_0; \chi) = \chi (r_0/r)^{-} D_{N+1}^{\chi R}(t, \chi r, \chi r_0) \tag{48}$$

where $\bar{D}^{\chi R}_{B+1} = (1+\partial_t)D^{\chi R}_{N+1}$. Expression (48) together with (44) and (45) constitute the explicit solution for shell sources. For a given source profile $\mathscr{I}(t)$, expression (48) must be convoluted with $\mathscr{I}(t)$.

The case of a *semi-infinite medium* is included in (48). Let τ (or τ_0) denote the optical depth of the observer (or source) below the boundary at R, i.e. $r = R - \tau$ and $r_0 = R - \tau_0$. We substitute these expressions for r and r_0 into (48) and go to the limit of R approaching infinity. It follows that

$$\begin{aligned} W_N(t, \tau, \tau_0; \chi) &= \chi(\tfrac{1}{2})^N \,\mathrm{e}^{-t} \sum_{\nu=1}^{N} \frac{EV(\nu-1)}{[((\nu-1)/2)!]^2} \frac{t^{N-\nu}}{(N-\nu)!} \\ &\quad\times \left(\frac{t^2 - \chi^2(\tau-\tau_0)^2}{4}\right)^{(\nu-1)/2} \theta(t - \chi|\tau - \tau_0|) \\ &\quad -\chi(\tfrac{1}{2})^N \,\mathrm{e}^{-t} \sum_{\nu=2}^{N\geqslant 2} \frac{EV(\nu-2)}{((\nu-2)/2)!(\nu/2)!} \frac{t^{N-\nu}}{(N-\nu)!} \left(\frac{t^2 - \chi^2(\tau+\tau_0)^2}{4}\right)^{(\nu-2)/2} \\ &\quad\times \left(\frac{t - \chi(\tau+\tau_0)}{2}\right) \theta[t - \chi(\tau+\tau_0)]. \end{aligned} \tag{49}$$

In order to generalise (47) we divide the right-hand side of (48) by $4\pi r_0^2$ and go to the limit of r_0 approaching zero. This leads to the following result for point sources:

$$W^R_N(t, r; \chi) = -\frac{\chi}{2\pi r}\frac{\partial}{\partial r}\Delta^{\chi R}_N(t, \chi r). \tag{50}$$

A thorough quantitative discussion of (48)-(50) and a comparison with results from Monte Carlo simulations for the same geometries will be presented in a separate paper by Schultz and Schweizer (1987). The following remarks are intended to give the reader a first idea about the applicability of the results derived in this paper.

(i) We have, so far, studied Gaussian-type sources with $\mathscr{I}(t) = \exp(-\alpha t^2)$. The parameters χ_0 through χ_8 given in § 2 are reasonable for all three geometries discussed in this paper and they correct the timing of the low scattering orders P_0 through P_8. Changing a characteristic parameter from $\sqrt{3}$ to χ_j shifts not only the position of the maximum of P_j but also alters its magnitude. The profiles $P_j(t; \chi_j)$ are therefore rescaled with appropriate multiplicative constants such that the magnitudes of the maxima are conserved.

(ii) After adjusting P_0 through P_8 in this manner good agreement with Monte Carlo simulations is obtained. As it turns out the analytical approach works best for the case of semi-infinite geometries. In this case, the agreement with numerical simulations extends down to small optical depths of order 0.1. In addition, the analytical method works also for the moderate Knudsen regime, i.e. for sources with $\alpha \gtrsim 1$.

(iii) The limitations of the analytical approach are most likely to become apparent if the dimension of the source is less than unity or point-like as in (13). For such sources, the medium must be clearly optically thick and the parameter α must be restricted to values less than unity. For instance, if the optical depth is of order unity the disagreement between the two approaches is not yet dramatic but clearly visible.

(iv) The pronounced precursors, predicted by (1) and (2) for $\alpha \gg 1$ and discussed in Schweizer (1985b), have disappeared in their original form. For $\alpha \gg 1$, the numerical simulations produce precursors strikingly similar to the type II precursors in Schweizer (1985b). However, Monte Carlo places the precursors right at the arrival time of the free radiation field and not at $t_p = \sqrt{3}\tau t_T$, as predicted by (1) and (2). It was pointed out above that the random walk approach in this paper does not apply to the Knudsen regime for point-like sources. The 'true precursors' can, for this reason, not be produced with our analytical method.

In the optically thick case, the numerical simulations provide essentially the same results if the scatterings are assumed to be perfectly isotropic. This explains in part the agreement we found between the two approaches and also why the analytical results do not apply to the optically thin case. Optically thin geometries can be treated effectively by Monte Carlo methods and this limitation of the random walk approach is, for this reason, not disturbing.

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Appendix 1

Here we show how the random walk approach presented in this paper has to be modified for the case of an absorptive medium. The starting point is a set of generalised diffusion equations given by

$$\partial N/\partial t + \nabla \cdot \boldsymbol{J} = -\kappa_a N + S \tag{A1.1}$$

$$(\kappa_T + \kappa_a)\boldsymbol{J} + \frac{\partial}{\partial t}\boldsymbol{J} + \tfrac{1}{3}\nabla N = 0 \tag{A1.2}$$

where κ_a denotes an absorption coefficient. Allowing for absorption processes such as (f–f) bremsstrahlung, it is straightforward to derive (A1.1) and (A1.2) along the lines given in § 2 in Schweizer (1985b). Again, we introduce dimensionless times and lengths $t \equiv t \cdot \kappa_T$, and $\boldsymbol{x} \equiv \boldsymbol{x} \cdot \kappa_T$, and we define dimensionless opacities

$$\alpha \equiv \kappa_a/\kappa_T \qquad \text{and} \qquad \hat{\kappa} = 1 + \alpha. \tag{A1.3}$$

The separation of the variables N and $\boldsymbol{J}$ yields the following generalised telegraphy equation:

$$\mathscr{L}N = (\hat{\kappa} + \alpha)\frac{\partial}{\partial t}N + \hat{\kappa}\alpha N + \frac{\partial^2}{\partial t^2}N - \tfrac{1}{3}\Delta N = \hat{\kappa}S + \frac{\partial}{\partial t}S. \tag{A1.4}$$

For the case of an infinite medium, the solution of (A1.4) is, modulo homogeneous solutions, given by $N = \bar{G} * S$, where now

$$\bar{G} = (\hat{\kappa} + \partial_t)G$$

and

$$G = \frac{1}{(2\pi)^4}\int d\omega \int d^3p \frac{\exp(-i\omega t)\exp(-i\boldsymbol{p}\cdot\boldsymbol{x})}{\frac{1}{3}\boldsymbol{p}^2 - \omega^2 - i\omega(\hat{\kappa}+\alpha) + \hat{\kappa}\alpha}. \tag{A1.5}$$

Repeating the arguments of § 2, we find again that $\bar{G} = \Sigma_{N=1}^{\infty} \bar{W}_N$, where $\bar{W}_N$ is an N-fold convolution product as in (7) but with $\bar{W}_1 = (\hat{\kappa} + \partial_t) W_1$ and

$$W_1 = \frac{\chi^2}{4\pi r} e^{-\hat{\kappa}t}\delta(t - \chi r). \tag{A1.6}$$

Note that the normalisation of the measures $\bar{W}_N$ is given by $\hat{\kappa}^{-N} \le 1$, where equality holds for the non-absorptive case, and that (14) is valid.

Turning to the case of a finite spherical geometry we see that the only modification of (18) or (24) is given by replacing $\exp(-t)$ in front of the multiple integral by $\exp(-\hat{\kappa}t)$. This follows from

$$(\hat{\kappa}+\partial_t)^N\, \mathrm{e}^{-\hat{\kappa}t} f = \mathrm{e}^{-\kappa t}(\partial/\partial t)^N f. \tag{A1.7}$$

Furthermore, one has to replace $(1-\mathrm{i}\omega)$ consistently by $\kappa \equiv \hat{\kappa} - \mathrm{i}\omega$. The solution of the integral equation remains unchanged and is given by (32) for the new definition of κ. The same is true for (37). The standard integrals (38) are discussed in appendix 2. From here it is easy to see that our results generalise to the case of an absorptive medium if the damping factor $\exp(-t)$, multiplying every individual scattering order, is consistently replaced by $\exp(-\hat{\kappa}t)$. The effects of absorption combined with multiple Compton scattering will be discussed in the third paper of this series.

Appendix 2

In this appendix, we derive (39) for the general case of an absorptive medium with $\kappa = \hat{\kappa} - \mathrm{i}\omega$. As pointed out by Nagel and Mészáros, it is advantageous to write the integral (38) in the form $(\hat{\kappa}+\partial_t)\hat{J}(M, Y)$, where

$$\hat{J}(M, Y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega t}\left(\frac{1-q}{1+q}\right)^M \frac{\mathrm{e}^{-\kappa q Y}}{2q\kappa}. \tag{A2.1}$$

As ω approaches $\pm\infty$, the function $q = q(\omega)$ converges to unity and the integrand in (A2.1) can be approximated by

$$\frac{1}{2\pi}\,\mathrm{e}^{-\hat{\kappa}Y}\frac{(1-q)^M}{2^{M+1}}\frac{\mathrm{e}^{-\mathrm{i}\omega[t-Y]}}{-\mathrm{i}\omega}.$$

For $t < Y$, the contour can be closed in the upper half-plane and the integral is equal to zero. For $t > Y$, the contour can be closed in the lower half-plane where we have the two poles $\mathrm{i}(\varepsilon - \hat{\kappa})$ and $-\mathrm{i}\hat{\kappa}$. We deform the contour into a narrow ellipse around the branch cut. The parametrisation of the integration path is given by

$$\omega(\lambda) = -\mathrm{i}[\hat{\kappa} - \varepsilon/2] - \mathrm{i}\tfrac{1}{2}\varepsilon\cos(\lambda + \mathrm{i}\delta) \tag{A2.2}$$

where $0 < \delta \ll 1$ and $\lambda \in [0, 2\pi]$. It follows from this that $\mathrm{d}\omega = q\kappa\, \mathrm{d}\lambda$. Furthermore, the total exponent appearing in the integrand of (A2.1) can conveniently be written as

$$-(\hat{\kappa} - \tfrac{1}{2}\varepsilon)t - \tfrac{1}{2}\varepsilon(t^2 - Y^2)^{1/2}\cos(\lambda + \mathrm{i}\delta - \mathrm{i}\gamma)$$

where $\exp(\gamma) = [t + Y/t - Y]^{1/2}$. Since

$$\left(\frac{1-q}{1+q}\right) = -\exp[\mathrm{i}(\lambda + \mathrm{i}\delta)] \tag{A2.3}$$

we find that

$$\hat{J}(M, Y) = \theta(t-Y)(-1)^M \tfrac{1}{2}\exp(-[\hat{\kappa} - \tfrac{1}{2}\varepsilon t])$$
$$\times \int_0^{2\pi} \frac{\mathrm{d}\lambda}{2\pi}\, \mathrm{e}^{\mathrm{i}M\lambda} \exp[-\tfrac{1}{2}\varepsilon(t^2 - Y^2)^{1/2}\cos(\lambda + \mathrm{i}\delta - \mathrm{i}\gamma)]. \tag{A2.4}$$

Integrating counterclockwise from $(0, 0)$ to $(2\pi, 0)$ to $(2\pi, \mathrm{i}\gamma)$ to $(0, \mathrm{i}\gamma)$ and back to $(0, 0)$, we see that the remaining integral in (A2.4) is the same as

$$\exp(-M\gamma)\int_0^{2\pi}\frac{\mathrm{d}\lambda}{2\pi}\exp\left(-\tfrac{1}{2}\varepsilon(t^2-Y^2)^{1/2}\cos\lambda\right)\mathrm{e}^{-\mathrm{i}M\lambda}.$$

This last integral can be given explicitly in terms of modified Bessel functions (see 9.6.19 in Abramowitz and Stegun (1970)). Altogether we have

$$\hat{J}(M, Y)=\tfrac{1}{2}\theta(t-Y)\exp[-(\hat{\kappa}-\tfrac{1}{2}\varepsilon)t]\left(\frac{t-Y}{t+Y}\right)^{M/2} I_M(\tfrac{1}{2}\varepsilon(t^2-Y^2)^{1/2}) \tag{A2.5}$$

Notice that the only modification of (39) due to absorption appears in the time exponent where $(1-\frac{1}{2}\varepsilon)$ has been replaced by $(\hat{\kappa}-\frac{1}{2}\varepsilon)$.

References

Abramowitz M and Stegun I A 1970 *Handbook of Mathematical Functions* (New York: Dover)
Hoffman J A, Lewin W H G, Doty J, Jernigan J G, Haney M and Richardson J A 1978 *Astrophys. J. Lett.* **221** 57
Lewin W H G, Vacca W D and Basinska E M 1984 *Astrophys. J. Lett.* **277** 57
Minin I N 1971 *Sov. Astron.* **15** 260
Nagel W and Mészáros P 1985 *J. Quant. Spectrosc. Radiat. Transfer* **34** 493
Schultz A L and Schweizer M A 1987 *J. Phys. A: Math. Gen.* submitted
Schweizer M A 1984a *J. Phys. A: Math. Gen.* **17** 2859
—— 1984b *Astrophys. J.* **280** 809
—— 1985a *Can. J. Phys.* **63** 956
—— 1985b *Astron. Astrophys.* **151** 79
Sobolev V V 1963 *A Treatise on Radiative Transfer* (New York: Van Nostrand)
Tawara Y *et al* 1984 *Astrophys. J. Lett.* **276** 41